Effective Boson-Fermion Dynamics for Subfermion Models

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Effective composite particle dynamics can be derived by weak mapping of quantum fields. This method was already applied to derive effective boson or boson-fermion coupling theories from a nonlinear subfermion field. In this paper we present an extension of those calculations to the general group theoretical treatment of two-fermion bound states and their coupling to (elementary) fermions within an arbitrary nonlinear spinor-isospinor field model. The resulting effective field equations are compared with the corresponding phenomenological expressions which for example underly the standard electroweak theory.

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Introduction

The derivation of an effective dynamics for composite particles (fields) is an essential problem common to all microscopic theories. Starting from a microscopic subdynamics the formation of composites and their mutual interaction emerge in effective equations that govern the system on a more macroscopic level. In relativistic quantum field theory effective actions are usually derived by evaluation of path integrals where composite structures are introduced by field operator products ('strong mapping'). However as it was shown in [1, 2] the strong mapping approach leads to conceptional difficulties apart from mathematical problems concerning the path integral.

Based on the algebraic properties of quantum fields Stumpf and coworkers have developed an alternative method to avoid the drawbacks of the conventional treatment. In this method the subtheory is formulated in terms of functional equations and states. Then by definition of composite particle states these functional equations and states can be transformed to a representation corresponding to the effective composite particle dynamics. For a detailed introduction see [1, 2].

This (weak) mapping procedure was successfully applied to derive effective dynamics of bound states within the subfermion model of Stumpf [3–7] which is based on Heisenberg's nonlinear spinor theory [8].

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The main concept was introduced in [4, 9, 10] considering the example of an effective boson-fermion coupling theory. Subsequently formfactors appearing in the composite particle description were considered [11]. An effective $SU(2)$ Yang-Mills theory with composite vector bosons was first derived in [5] and then confirmed in [12, 13]. Based on these considerations models of composite electroweak bosons, leptons, Han-Nambu quarks and gluons were presented [6, 14]. Together with the inclusion of gravity [7] these investigations should be completed to a unified subfermion model of elementary particles.

The present investigation is mainly devoted to the question of the possible explanation of the phenomenological theories that underly the standard model before symmetry breaking. On the subfermion level we consider a nonlinear spinor-isospinor field model with global $SU(2) \times U(1)$ isospin symmetry. Based on this model the simplest way to derive the principle structure of the standard model of electroweak interactions on the composite particle level is to represent the fermions by the elementary spinor fields and the bosons (gauge bosons and possible composite scalar bosons) by two-fermion bound states. (In a realistic subfermion model of electroweak interactions quarks and leptons are expected to be composed of at least three subfermions.) For this model we will treat the most general case of the effective dynamics using the complete algebraic spectrum of two-particle bound states (scalars and vectors within isospin 0 and 1 and fermion number 0 and ± 2). The resulting effective field equations will then be compared with the

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corresponding phenomenological equations. In this way the afore mentioned investigations are generalized and completed by the group theoretical classification of all possible effective theories resulting from a group theoretical analysis of bound two-fermion states.

In forthcoming papers more realistic extensions will be presented including three-particle fermion states and isospin symmetry breaking. Thus a general structural analysis of possible subfermionic explanations of the standard model will be given. As our emphasis lies on the description of the formation of dynamical symmetries on the composite particle level our treatment may be regarded as complementary to earlier investigations of Dürr and Saller [15, 16] where the reduction of effective symmetries (of the standard model) to fundamental ones (of a subfermion model) was considered.

1. Nonlinear Spinor Field Model

In this section we introduce the general formulation of a nonlinear spinor field model with local four-fermion interaction as described in [17]. We consider a Dirac-spinor-isospinor field $\psi_{\alpha A}(x)$ with Dirac index $\alpha=1,\ldots,4$ and isospin index $A=1,2$. The self-interaction is supposed to be Lorentz- and isospin-$U(2)$-invariant. The antisymmetrization of the (normal ordered) vertex is explicitly taken into account by an antisymmetrized Fierz expansion with characteristic coupling constants for a special interaction model. With this general Fierz expansion the field equation of the spinor-isospinor field $\psi(x)$ reads

$$(i\gamma^\mu\partial_\mu-m_0)_{\text{reg}}\psi(x)=\sum_h g_h v_h\psi(x)(\bar\psi(x)v^h\psi(x))+\sum_h \bar g_h v_h G\bar\psi^T(x)(\psi^T(x)Gv^h\psi(x)) \tag{1}$$

with the basic elements

$$\begin{aligned}v^h=v^{st}&\equiv\Gamma^s\otimes\tau^t\in\{\gamma^\mu,\Sigma^{\mu\nu},\gamma_5\gamma^\mu,i\gamma_5,\mathbb{1}\}\otimes\{\mathbb{1},\tau^k\},\\ v_h=v_{st}&\equiv\Gamma_s\otimes\tau^t\in\{\gamma_\mu,\Sigma_{\mu\nu},\gamma_5\gamma_\mu,i\gamma_5,\mathbb{1}\}\otimes\{\mathbb{1},\tau^k\},\\ h&=(s,t),\end{aligned} \tag{2}$$

in the direct product space of spin and isospin and corresponding coupling constants g_h and $\bar g_h$. The Pauli matrices in isospace are denoted by $\{\tau^t, t=0,\ldots,3\}$ with $\tau^0:=\mathbb{1}$. Further the abbreviation $G:=C\otimes c_\tau$, $c_\tau:=-i\tau^2$ is used. The kinetic operator contains a mass parameter m_0; the suffix 'reg' indicates some kind of regularization to achieve a regular theory. In the subfermion model of Stumpf for this purpose a nonperturbative auxiliary field regularization is used. In the following we don't use this kind of regularization explicitly. Rather we suppose the theory to be sufficiently regular for the further evaluation.

The field equation (1) and the corresponding charge conjugated equation can be compactly written by the introduction of a superspinor $\boldsymbol\psi=(\boldsymbol\psi_\Lambda)$, $\Lambda=1,2$ which combines the spinor ψ and the charge conjugated spinor $\psi^G:=G\bar\psi^T=Cc_\tau\bar\psi^T$ by the definitions

$$\begin{aligned}\boldsymbol\psi_{\Lambda=1}&:=\psi,\\ \boldsymbol\psi_{\Lambda=2}&:=\psi^G.\end{aligned} \tag{3}$$

Denoting Pauli matrices in superspace by $\{\lambda^l, l=0,\ldots,3\}$ the field equation (1) and its charge conjugated counterpart can then be combined in the following way:

$$\begin{aligned}(i\gamma^\mu\partial_\mu-m_0)_{\text{reg}}\boldsymbol\psi
&=\frac12\sum_h g_h v_h[\boldsymbol\psi(\boldsymbol\psi^T Gv^h c_\lambda\lambda^0\boldsymbol\psi)+\lambda^3\boldsymbol\psi(\boldsymbol\psi^T Gv^h c_\lambda\lambda^3\boldsymbol\psi)]\\
&-\frac12\sum_h \bar g_h v_h[\lambda^1\boldsymbol\psi(\boldsymbol\psi^T Gv^h c_\lambda\lambda^1\boldsymbol\psi)\\
&\qquad+\lambda^2\boldsymbol\psi(\boldsymbol\psi^T Gv^h c_\lambda\lambda^2\boldsymbol\psi)]\end{aligned} \tag{4}$$

(with $c_\lambda:=-i\lambda^2$). This equation can be expressed in the more compact form

$$K\boldsymbol\psi(x)=\frac12\sum_H g_H\boldsymbol v_H\boldsymbol\psi(x)(\boldsymbol\psi^T(x)\boldsymbol w^H\boldsymbol\psi(x)),\quad H=(h,l)=(s,t,l), \tag{5}$$

where the definition of the kinetic operator K is obvious and the vertex is characterized by the elements

$$\begin{aligned}\boldsymbol v_H=\boldsymbol v_{stl}&:=v_h\lambda^l=\Gamma_s\tau^t\lambda^l,\\ \boldsymbol w^H=\boldsymbol w^{stl}&:=Cc_\tau c_\lambda\boldsymbol v^{stl}=C\Gamma^s c_\tau\tau^t c_\lambda\lambda^l\end{aligned} \tag{6}$$

with the corresponding coupling constants

$$g_{st0}=g_{st3}:=g_{st},\quad g_{st1}=g_{st2}:=-\bar g_{st}. \tag{7}$$

In a superindex notation with $I=(Z,x)=(\alpha,A,\Lambda,x)$ and an extension of the summation rule to integrations for continuous indices the field equations (5) read

$$K_{I_1I}\boldsymbol\psi_I=V_{I_1I_2I_3I_4}\boldsymbol\psi_{I_2}\boldsymbol\psi_{I_3}\boldsymbol\psi_{I_4} \tag{8}$$

with

$$\begin{aligned}K_{I_1I_2}&:=K_{Z_1Z_2}(x_1)\delta(x_1-x_2)\\
&:=(D^\mu\partial_\mu^{(1)}-M)_{Z_1Z_2/\text{reg}}\delta(x_1-x_2),\\
D^\mu_{Z_1Z_2}&:=i\gamma^\mu_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2},\\
M_{Z_1Z_2}&:=m_0\delta_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}\end{aligned} \tag{9}$$

and

$$V_{I_1I_2I_3I_4} := V_{Z_1Z_2Z_3Z_4}\,\delta(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_1-x_4),$$
$$V_{Z_1Z_2Z_3Z_4} := \frac{1}{2}\sum_H g_H(v_H)_{Z_1Z_2}(w^H)_{Z_3Z_4}. \tag{10}$$

With these field equations together with the anticommutation relations $\{\psi_I, \psi_{I'}\} = A_{II'}$ the spinor quantum field theory is defined.

As described in [2] the quantum states $\{|a\rangle\}$ of the field theory can be characterized by normal ordered matrix elements $\varphi_n(I_1 \dots I_n|a) = \langle\Omega|\mathcal{N}\{\psi_{I_1}\dots\psi_{I_n}\}|a\rangle$ respectively by the corresponding generating functional

$$\mathcal{F}[j,a] = \sum_{n=0}^{\infty} \frac{i^n}{n!}\varphi_n(I_1 \dots I_n|a)\, j_{I_1}\dots j_{I_n} \tag{11}$$

with anticommuting sources j_I. In functional quantum theory according to [18, 19, 20] additionally an auxiliary functional Fock space is constructed with a functional vacuum state $|0\rangle$ and the sources j_I acting as creation operators. In this case the functional derivatives $\partial_I := \delta/\delta j_I$ can be regarded as (anticommuting) destruction operators with $\{j_I, \partial_{I'}\} = \delta_{II'}$ and the φ_n-functions can be obtained from the corresponding functional state $|\mathcal{F}[j,a]\rangle := \mathcal{F}[j,a]|0\rangle$ by a projection with $(\frac{1}{i})^n\langle 0|\partial_{I_n}\dots\partial_{I_1}$. For details of this functional formulation see [18–20].

Using the field equations and the anticommutation relations a functional equation for $|\mathcal{F}[j,a]\rangle$ can be derived. For the weak mapping procedure this functional equation is considered for equal times $t_1 = \dots = t_n = t$ and formulated in an energy representation exploiting the symmetry condition corresponding to energy: $P_0|a\rangle = E_a|a\rangle$. This functional energy equation is given by [2]

2. Effective Boson-Fermion Coupling Theory

Equation (12) can be written in the compact form

$$(E_a - E_0)|\mathcal{F}[j,a]\rangle_{/t} = \mathcal{H}\left[j, \frac{\delta}{\delta j}\right]|\mathcal{F}[j,a]\rangle_{/t} \tag{14}$$

with a functional Hamiltonian $\mathcal{H}\left[j, \frac{\delta}{\delta j}\right]$ which completely characterizes the spinor quantum field theory. The derivation of composite particle dynamics by weak mapping starting from (14) is generally described in [1, 2]. As pointed out in the introduction here we are interested in a combined boson-fermion dynamics with bosons as two-particle bound states and fermions represented by the elementary spinor field. This problem was already discussed in [21] using a short-cut calculation technique developped in [4, 9–11]. In this paper we will adopt this technique which is suitable in a low energy approximation where exchange forces are neglected.

In order to derive a composite particle dynamics at first the composite particle states have to be introduced. According to [2] these composite particle states are defined by the solutions of the particle number conserving diagonal part of (12). For the two-particle sector the complete set of these solutions can be characterized by a bosonic index K and is denoted by $\{\mathcal{C}_K^{I_1I_2}\}$. To perform the weak mapping additionally dual states $\{\mathcal{R}^K_{I_1I_2}\}$ have to be introduced with the orthogonality relations

$$\mathcal{R}^K_{I_1I_2}\mathcal{C}_{K'}^{I_1I_2} = \delta^K{}_{K'} \tag{15}$$

and the completeness relation

$$\mathcal{R}^K_{I_1I_2}\mathcal{C}_K^{I'_1I'_2} = \delta^{[I'_1}{}_{[I_1}\delta^{I'_2]}{}_{I_2]}. \tag{16}$$

$$\begin{aligned}(E_a - E_0)|\mathcal{F}[j,a]\rangle_{/t} = j_{I_1}\{&\tilde{K}_{I_1I_2}\partial_{I_2} + W_{I_1I_2I_3I_4}[\partial_{I_4}\partial_{I_3}\partial_{I_2} - 3F_{I_4I'_4}j_{I'_4}\partial_{I_3}\partial_{I_2}\\ &+ (3F_{I_4I'_4}F_{I_3I'_3} + \tfrac{1}{4}A_{I_4I'_4}A_{I_3I'_3})j_{I'_4}j_{I'_3}\partial_{I_2}\\ &- (F_{I_4I'_4}F_{I_3I'_3} + \tfrac{1}{4}A_{I_4I'_4}A_{I_3I'_3})F_{I_2I'_2}j_{I'_4}j_{I'_3}j_{I'_2}]\}|\mathcal{F}[j,a]\rangle_{/t}\end{aligned} \tag{12}$$

with $I = (Z, \boldsymbol{x})$ and

$$\tilde{K}_{I_1I_2} := (\Gamma^k\partial_k^1 + M)_{Z_1Z_2}\,\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2),\quad \Gamma^k\partial_k + M := -\gamma^0(i\gamma^k\partial_k - m_0),$$
$$W_{I_1I_2I_3I_4} := W_{Z_1Z_2Z_3Z_4}\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_3)\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_4),$$
$$W_{Z_1Z_2Z_3Z_4} := \frac{1}{2}\sum_H g_H(\boldsymbol{u}_H)_{Z_1Z_2}\boldsymbol{w}^H_{Z_3Z_4},\quad \boldsymbol{u}_H := -\gamma^0\boldsymbol{v}_H. \tag{13}$$

$F_{I_1I_2} := F_{Z_1Z_2}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ denotes the equal time limit of the propagator, and E_0 the ground state energy.

By means of this set bosonic sources

$$b_K = \mathscr{C}_K^{I_1 I_2} j_{I_1} j_{I_2} \tag{17}$$

can be defined with the inversion

$$j_{I_1} j_{I_2} = R_{I_1 I_2}^K b_K, \tag{18}$$

and the functional states (11) can be reformulated as boson-fermion functional statets:

$$\mathscr{F}[j,a]\rangle_{/t} = \sum_{m,n=0}^{\infty} \frac{i^m}{m!} \frac{i^n}{n!} \varrho(K_1 \dots K_m I_1 \dots I_n | a)_{/t} \cdot b_{K_1} \dots b_{K_m} j_{I_1} \dots j_{I_n} |0\rangle =: |\tilde{\mathscr{F}}[j,b;a]\rangle_{/t}. \tag{19}$$

Then, using the functional chain rule

$$\begin{aligned}\frac{\delta}{\delta j_I} |\mathscr{F}[j]\rangle_{/t} &= \frac{\delta}{\delta j_I} |\tilde{\mathscr{F}}[j, b(j)]\rangle_{/t} \\ &= \left[\sum_K \left(\frac{\delta}{\delta j_I} b_K(j) \right) \frac{\delta}{\delta b_K} + \frac{\delta}{\delta j_I} \right] |\tilde{\mathscr{F}}[j,b]\rangle_{/t} \\ &= \left[\sum_K 2 \mathscr{C}_K^{II'} j_{I'} \frac{\delta}{\delta b_K} + \frac{\delta}{\delta j_I} \right] |\tilde{\mathscr{F}}[j,b]\rangle_{/t},\end{aligned} \tag{20}$$

from (12) the effective boson-fermion functional equation

$$\tilde{\mathscr{H}}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b} \right] |\tilde{\mathscr{F}}[j,b;a]\rangle_{/t} = (E_a - E_0) |\tilde{\mathscr{F}}[j,b;a]\rangle_{/t} \tag{21}$$

can be derived. This derivation is described in detail in [4, 9, 10] and [12]. For the purpose of the present investigation it is sufficient to consider only the leading terms corresponding to a classical low energy approximation. In this case the effective functional Hamiltonian reads

$$\begin{aligned}&\tilde{\mathscr{H}}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b} \right] \\ &= [2 \mathscr{R}_{I_1 I}^{K_1} \tilde{K}_{I_1 I_2} \mathscr{C}_{K_2}^{I_2 I} + 6 W_{I_1 I_2 I_3 I_4} F_{I_2 I} \mathscr{R}_{I_1 I}^{K_1} \mathscr{C}_{K_2}^{I_4 I_3}] \cdot b_{K_1} \frac{\delta}{\delta b_{K_2}} \\ &\quad + 12 W_{I_1 I_2 I_3 I_4} \mathscr{R}_{I_1 I}^{K_3} \mathscr{C}_{K_1}^{I_3 I_4} \mathscr{C}_{K_2}^{I_2 I} b_{K_3} \frac{\delta}{\delta b_{K_2}} \frac{\delta}{\delta b_{K_1}} \\ &\quad + 6 W_{I_1 I_2 I_3 I_4} \mathscr{C}_K^{I_3 I_4} j_{I_1} \frac{\delta}{\delta j_{I_2}} \frac{\delta}{\delta b_K} \\ &\quad + 3 W_{I_1 I_2 I_3 I_4} F_{I_2 I} \mathscr{R}_{I_1 I}^K b_K \frac{\delta}{\delta j_{I_3}} \frac{\delta}{\delta j_{I_4}} \\ &\quad + \tilde{K}_{I_1 I_2} j_{I_1} \frac{\delta}{\delta j_{I_2}} + W_{I_1 I_2 I_3 I_4} j_{I_1} \frac{\delta}{\delta j_{I_4}} \frac{\delta}{\delta j_{I_3}} \frac{\delta}{\delta j_{I_2}} \\ &:= \mathscr{H}_b\left[b, \frac{\delta}{\delta b} \right] + \mathscr{H}_{b-f}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b} \right] + \mathscr{H}_f\left[j, \frac{\delta}{\delta j} \right].\end{aligned} \tag{22}$$

As can be seen from (22), $\tilde{\mathscr{H}}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b} \right]$ splits into three terms corresponding to the bosonic, fermionic, and the boson-fermion coupling part of the effective boson-fermion theory.

On the functional level the effective theory is described by the functional equation (21) with $\tilde{\mathscr{H}}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b} \right]$ from (22) and $|\tilde{\mathscr{F}}[j,b;a]\rangle_{/t}$ according to (19). However it is convenient to re-express the effective functional equation in terms of a field operator theory with corresponding field equations. For this purpose the functions $\varrho(I_1 \dots I_n K_1 \dots K_m | a)_{/t}$ are considered as matrix elements of (antisymmetrized) products of fermionic fields $\psi_I(t)$ (with $I = (Z, \boldsymbol{x})$) and (symmetrized) products of bosonic fields $\Phi^K(t)$:

$$\varrho(I_1 \dots I_n K_1 \dots K_m | a)_{/t} =: \langle \Omega | \psi_{I_1}(t) \dots \psi_{I_n}(t) \, \Phi^{K_1}(t) \dots \Phi^{K_m}(t) | a \rangle. \tag{23}$$

Then the field equations corresponding to the effective theory result by functional projections with $\frac{1}{i} \langle 0 | \frac{\delta}{\delta j_I}$ and $\frac{1}{i} \langle 0 | \frac{\delta}{\delta b_K}$:

$$\frac{1}{i} \langle 0 | \frac{\delta}{\delta j_I} E_a |\tilde{\mathscr{F}}[j,b;a]\rangle_{/t} = E_a \langle \Omega | \psi_I(t) | a \rangle = i \partial_t \langle \Omega | \psi_I(t) | a \rangle, \tag{24}$$

$$\frac{1}{i} \langle 0 | \frac{\delta}{\delta b_K} E_a |\tilde{\mathscr{F}}[j,b;a]\rangle_{/t} = E_a \langle \Omega | \Phi^K(t) | a \rangle = i \partial_t \langle \Omega | \Phi^K(t) | a \rangle. \tag{25}$$

Applying these projections to (21) with $\tilde{\mathscr{H}}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b} \right]$ from (22), it results from (24)

$$\begin{aligned}&i \partial_t \langle \Omega | \psi_{I_1}(t) | a \rangle \\ &= \tilde{K}_{I_1 I_2} \langle \Omega | \psi_{I_2}(t) | a \rangle \\ &\quad - W_{I_1 I_2 I_3 I_4} \langle \Omega | \psi_{I_2}(t) \, \psi_{I_3}(t) \, \psi_{I_4}(t) | a \rangle \\ &\quad + 6 i W_{I_1 I_2 I_3 I_4} \mathscr{C}_K^{I_3 I_4} \langle \Omega | \psi_{I_2}(t) \, \Phi^K(t) | a \rangle\end{aligned} \tag{26}$$

and from (25)

$$\begin{aligned}&i \partial_0 \langle \Omega | \Phi^{K_1}(t) | a \rangle \\ &= [2 \mathscr{R}_{I_1 I}^{K_1} \tilde{K}_{I_1 I_2} \mathscr{C}_{K_2}^{I_2 I} + 6 W_{I_1 I_2 I_3 I_4} F_{I_2 I} \mathscr{R}_{I_1 I}^{K_1} \mathscr{C}_{K_2}^{I_4 I_3}] \\ &\quad \cdot \langle \Omega | \Phi^{K_2}(t) | a \rangle\end{aligned}$$

$$+12\, i\, W_{I_1 I_2 I_3 I_4} \mathscr{R}^{K_1}_{I_1' I} \mathscr{C}^{I_3 I_4}_{K_2} \mathscr{C}^{I_2 I}_{K_3} \langle \Omega | \Phi^{K_2}(t)\, \Phi^{K_3}(t) | a \rangle$$
$$+3\, i\, W_{I_1 I_2 I_3 I_4} F_{I_2 I} \mathscr{R}^{K_1}_{I_1' I} \langle \Omega | \psi_{I_3}(t)\, \psi_{I_4}(t) | a \rangle. \qquad (27)$$

Equations (26) and (27) are the effective field equations for (the matrix elements of) the spinor field ψ_I and the bosonic fields Φ^K (corresponding to the $\mathscr{C}_K$). For the further evaluation the explicit expressions for the bosonic states $\mathscr{C}^{I_1 I_2}_K$ and their duals $\mathscr{R}^K_{I_1 I_2}$ have to be given.

3. Composite Particle States

As mentioned above the bosonic states are defined by the solutions of the diagonal part of the functional equation. The calculation of these states is extensively described in [22] and [17]. In the following, however, we will not use the exact solutions. Rather we make suitable approximations concerning the orbital part of the wave functions. This procedure was already performed in [5] and confirmed by exact calculations in [12]. Additionally we reduce the complete set of solutions which contains bound states as well as scattering states to the subset of bound states. Furthermore in a low energe approximation we consider only ground states without internal excitations (leading to a pure algebraic treatment). Then the bosonic states with momentum $\boldsymbol{k}$ are given by

$$\mathscr{C}^K_{I_1 I_2} = \mathscr{C}^H_{Z_1 Z_2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\, \chi(\boldsymbol{u}|\boldsymbol{k}) = \mathscr{C}^{stl}_{Z_1 Z_2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\, \chi(\boldsymbol{u}|\boldsymbol{k})$$
$$= \mathscr{S}^s_{\alpha_1 \alpha_2}\, \mathscr{T}^t_{A_1 A_2}\, \mathscr{L}^l_{A_1 A_2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\, \chi(\boldsymbol{u}|\boldsymbol{k}) \qquad (28)$$

with the classification index $K = (H, \boldsymbol{k}) = (s, t, l, \boldsymbol{k})$. The matrices $\mathscr{S}^s$, $\mathscr{T}^t$, and $\mathscr{L}^l$ refer to spin, isospin, and superspin. Explicitly the basis sets are given by

$$\{\mathscr{S}^s;\ s = 1, \ldots, 16\} = \{(\Gamma_s)^+ C\}$$
$$= \{\gamma^\mu C,\ \Sigma^{\mu\nu} C,\ \gamma^\mu \gamma_5 C,\ -i\gamma_5 C,\ C\}$$
$$= \{\gamma^\mu C,\ \Sigma^{0k} C,\ \Sigma^k C,\ \gamma^\mu \gamma_5 C,\ -i\gamma_5 C,\ C\} \qquad (29)$$

(with $\Sigma^k = \frac{1}{2}\varepsilon_{ijk}\Sigma^{ij} = \gamma_5 \gamma^0 \gamma^k$) for spin,

$$\{\mathscr{T}^t = \tau^t c_\tau,\ t = 0, 1, 2, 3\} \qquad (30)$$

(with $c_\tau = -i\tau^2$) for isospin, and

$$\{\mathscr{L}^l = \lambda^l c_\lambda,\ l = 0, 1, 2, 3\} \qquad (31)$$

(with $c_\lambda = -i\lambda^2$) for superspin. The use of this basis yields the following relation to the notation (6) for the vertex elements: $\mathscr{C}^H = \mathscr{C}^{stl} = (\Gamma_s)^+ C \tau^t c_\tau \lambda^l c_\lambda = (\boldsymbol{v}_H)^+ C c_\tau c_\lambda = -(\boldsymbol{w}_H)^+$. From this follows the orthogonality relation

$$\mathrm{tr}\{\boldsymbol{w}_H \mathscr{C}^{H'}\} = -16\,\delta_H^{\ H'} \qquad (32)$$

and the relation

$$\mathscr{C}^H = \mathscr{C}^{stl} = \eta_s \boldsymbol{v}^{stl} C c_\tau c_\lambda$$

with

$$\eta_s = \begin{cases} 1 & \text{for} \quad \mathscr{S}^s \in \{\gamma^\mu, \Sigma^{\mu\nu}, \mathbb{1}\} \\ -1 & \phantom{\text{for}} \quad \mathscr{S}^s \in \{\gamma_5 \gamma^\mu, -i\gamma_5\} \end{cases}. \qquad (33)$$

These relation will be helpful for the evaluation of the effective dynamics.

The orbital part of the basis states (28) is given by a symmetric, regular wave function

$$\chi(-\boldsymbol{u}|\boldsymbol{k}) = \chi(\boldsymbol{u}|\boldsymbol{k}) \qquad (34)$$

with the low energy approximation

$$\chi(\boldsymbol{u}|\boldsymbol{k}) \approx \chi(\boldsymbol{u}|0) =: \chi(\boldsymbol{u}). \qquad (35)$$

Then, from the total antisymmetry of the states (28) follows the antisymmetry of the algebraic part $\mathscr{C}^H_{Z_1 Z_2}$.

The dual states, which are defined by the relations (15) and (16), are then given by

$$\mathscr{R}^{I_1 I_2}_K = \mathscr{R}^{Z_1 Z_2}_{stl}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{z}}\, \chi^*(\boldsymbol{u}|\boldsymbol{k})$$
$$= \hat{\mathscr{S}}^{\alpha_1 \alpha_2}_s\, \hat{\mathscr{T}}^{A_1 A_2}_t\, \hat{\mathscr{L}}^{A_1 A_2}_l\, e^{-i\boldsymbol{k}\cdot\boldsymbol{z}}\, \chi^*(\boldsymbol{u}|\boldsymbol{k}) \qquad (36)$$

with

$$\hat{\mathscr{S}}_s = \tfrac{1}{4}(\mathscr{S}^s)^*, \quad \hat{\mathscr{T}}_t = \tfrac{1}{2}(\mathscr{T}^t)^*, \quad \hat{\mathscr{L}}_l = \tfrac{1}{2}(\mathscr{L}^l)^*. \qquad (37)$$

I.e. the dual states are simply $\mathscr{R}^{I_1 I_2}_K = \frac{1}{16}(\mathscr{C}^K_{I_1 I_2})^*$. (This simple relation no longer holds if the auxiliary field regularization is explicitly considered.)

With respect to the boson fields we introduce the following explicit notations:

$$\{\Phi_K\} = \{\Phi_H(z) = \{\Phi_{stl}(z)\}$$
$$= \{A^{tl}_\mu(z),\ F^{tl}_{\mu\nu}(z),\ G^{tl}_\mu(z),\ \phi^{tl}(z),\ Z^{tl}(z)\} \qquad (38)$$
$$= \{A^{tl}_\mu(z),\ E^{tl}_k(z),\ B^{tl}_k(z),\ G^{tl}_\mu(z),\ \phi^{tl}(z),\ Z^{tl}(z)\}$$

corresponding to the following relations between the states $\mathscr{C}^H = \mathscr{S}^s \mathscr{T}^t \mathscr{L}^l$ and the fields Φ_H:

$$\begin{aligned}
\gamma^\mu C \tau^t c_\tau \lambda^l c_\lambda &=: \mathscr{S}^\mu \mathscr{T}^t \mathscr{L}^l \leftrightarrow A^{tl}_\mu, \\
\Sigma^{0k} C \tau^t c_\tau \lambda^l c_\lambda &=: \mathscr{S}^{0k} \mathscr{T}^t \mathscr{L}^l \leftrightarrow E^{tl}_k, \\
\Sigma^k C \tau^t c_\tau \lambda^l c_\lambda &=: \mathscr{S}^{50k} \mathscr{T}^t \mathscr{L}^l \leftrightarrow B^{tl}_k, \\
\gamma^\mu \gamma_5 C \tau^t c_\tau \lambda^l c_\lambda &=: \mathscr{S}^{\mu 5} \mathscr{T}^t \mathscr{L}^l \leftrightarrow G^{tl}_\mu, \\
-i\gamma_5 C \tau^t c_\tau \lambda^l c_\lambda &=: \mathscr{S}^5 \mathscr{T}^t \mathscr{L}^l \leftrightarrow \phi^{tl}, \\
C \tau^t c_\tau \lambda^l c_\lambda &=: \mathscr{S}^C \mathscr{T}^t \mathscr{L}^l \leftrightarrow Z^{tl}.
\end{aligned} \qquad (39)$$

In this notation the Lorentz properties of the boson fields are made manifest: A_μ: vector, $F_{\mu\nu}$: tensor, G_μ: axialvector, ϕ: pseudoscalar, Z: scalar. The isospin properties are characterized by the values of the index t: $t=0$: isosinglet, $t=1,2,3$: isotriplet. Finally the superspin index l marks the fermion number properties. $l=0$ stands for $(\psi\psi^G+\psi^G\psi)$-amplitudes, $l=1$ for $(\psi\psi-\psi^G\psi^G)$, $l=2$ for $\frac{1}{i}(\psi\psi+\psi^G\psi^G)$, and $l=3$ for $\psi\psi^G-\psi^G\psi)$. Due to the antisymmetry of the wave functions only those combinations (s, t, l) are allowed which correspond to antisymmetric $\mathcal{S}^s\mathcal{T}^t\mathcal{L}^l$. (Shortly: $(s\,t\,l)\in\mathcal{A}_s$.)

4. Evaluation of the Effective Boson Dynamics

Now we are able to evaluate the functional boson-fermion Hamiltonian (22). We start with the bosonic part, which is given by

$$\mathcal{H}_b=[2\mathcal{R}^{K_1}_{I_1 I}\tilde{K}_{I_1 I_2}\mathcal{C}^{I_2 I}_{K_2}+6W_{I_1 I_2 I_3 I_4}F_{I_2 I}\mathcal{R}^{K_1}_{I_1 I}\mathcal{C}^{I_4 I_3}_{K_2}]\,b_{K_1}\frac{\delta}{\delta b_{K_2}}+12W_{I_1 I_2 I_3 I_4}\mathcal{R}^{K_3}_{I_1 I}\mathcal{C}^{I_3 I_4}_{K_1}\mathcal{C}^{I_2 I}_{K_2}b_{K_3}\frac{\delta}{\delta b_{K_2}}\frac{\delta}{\delta b_{K_1}}$$
$$=:\mathcal{H}_{\mathrm{kin}}+\mathcal{H}_{bb}. \tag{40}$$

It contains a kinematic part $\mathcal{H}_{\mathrm{kin}}$ and an interaction part $\mathcal{H}_{bb}$ which describes the boson-boson coupling. The kinematic part of $\mathcal{H}_b$ is given by

$$\mathcal{H}_{\mathrm{kin}}=b_{K_1}[2\mathcal{R}^{K_1}_{I_1 I}\tilde{K}_{I_1 I_2}\mathcal{C}^{I_2 I}_{K_2}+6W_{I_1 I_2 I_3 I_4}F_{I_2 I}\mathcal{R}^{K_1}_{I_1 I}\mathcal{C}^{I_4 I_3}_{K_2}]\frac{\delta}{\delta b_{K_2}}$$
$$=:\mathcal{H}_0+\mathcal{H}_1. \tag{41}$$

Herein the part $\mathcal{H}_0$ is of purely kinematical nature (corresponding to the Bargmann-Wigner equations) while the second therm $\mathcal{H}_1$ leads to mass corrections.

We first consider the Bargmann-Wigner term $\mathcal{H}_0$. With $K=(H,\boldsymbol{k})=(s,t,l,\boldsymbol{k})$ and $I=(Z,\boldsymbol{x})$ the evaluation of the $\boldsymbol{x}_2$-integral in the I_2-summation and the low energy approximation (35) leads with the Fourier transforms of the sources

$$b_H(\boldsymbol{z})=\int \mathrm{d}^3k\, e^{-i\boldsymbol{k}\cdot\boldsymbol{z}}\, b_H(\boldsymbol{k}),\quad \frac{\delta}{\delta b_H}(\boldsymbol{z})=\int \mathrm{d}^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\frac{\delta}{\delta b_H}(\boldsymbol{k}) \tag{42}$$

to the expression

$$\mathcal{H}_0=2\int \mathrm{d}^3z\,\mathrm{d}^3u\, b_{H_1}(\boldsymbol{z})\,\chi^*(\boldsymbol{u})\,[\tfrac{1}{2}\mathrm{tr}\,\{(\mathcal{R}^{H_1})^T\Gamma^k\mathcal{C}_{H_2}\}\,\partial^{(z)}_k+\mathrm{tr}\,\{(\mathcal{R}^{H_1})^T\Gamma^k\mathcal{C}_{H_2}\}\,\partial^{(u)}_k$$
$$+\mathrm{tr}\,\{(\mathcal{R}^{H_1})^T M\mathcal{C}_{H_2}\}]\,\chi(\boldsymbol{u})\frac{\delta}{\delta b_{H_2}}(\boldsymbol{z}). \tag{43}$$

With the normalization of the wave functions $\int \mathrm{d}^3u\,\chi^*(\boldsymbol{u})\,\chi(\boldsymbol{u})=1$ and the symmetry property $\int \mathrm{d}^3u\,\chi^*(\boldsymbol{u})\cdot\partial^{(u)}_k\chi(\boldsymbol{u})\simeq 0$ for ground states one finds

$$\mathcal{H}_0\int \mathrm{d}^3z\, b^{H_1}(\boldsymbol{z})\,[\underbrace{\mathrm{tr}\,\{\mathcal{R}^T_{H_1}\Gamma^k\mathcal{C}^{H_2}\}}_{=:\mathrm{Tr}_1(H_1 H_2,k)}\partial_k+2\underbrace{\mathrm{tr}\,\{\mathcal{R}^T_{H_1}M\mathcal{C}^{H_2}\}}_{=:\mathrm{Tr}_2(H_1 H_2)}]\frac{\delta}{\delta b_{H_2}}(\boldsymbol{z}). \tag{44}$$

For the further evaluation we insert the expressions for Γ^k and M according to (13) and $\mathcal{C}^H$ and $\mathcal{R}_H=\frac{1}{16}(\mathcal{C}^H)^*$ according to (28) and (36) into the traces $\mathrm{Tr}_1(H_1 H_2,k)$ and $\mathrm{Tr}_2(H_1 H_2)$. The first trace yields

$$\mathrm{Tr}_1(H_1 H_2,k)=\mathrm{Tr}_1(s_1 t_1 l_1 s_2 t_2 l_2,k)$$
$$=-i\underbrace{\mathrm{tr}\,\{(\tfrac{1}{4}\mathcal{S}^{s_1})^+\gamma^0\gamma^k\mathcal{S}^{s_2}\}}_{=:\mathrm{Tr}_1(s_1,s_2,k)}\underbrace{\mathrm{tr}\,\{(\tfrac{1}{2}\mathcal{T}^{t_1})^+\mathcal{T}^{t_2}\}}_{=\delta^{t_1 t_2}}\underbrace{\mathrm{tr}\,\{(\tfrac{1}{2}\mathcal{L}^{l_1})^+\mathcal{L}^{l_2}\}}_{=\delta^{l_1 l_2}}, \tag{45}$$

and the second:

$$\mathrm{Tr}_2(H_1 H_2)=\mathrm{Tr}_2(s_1 t_1 l_1 s_2 t_2 l_2)=\underbrace{\mathrm{tr}\,\{(\tfrac{1}{4}\mathcal{S}^{s_1})^+\gamma^0\mathcal{S}^{s_2}\}}_{=:\mathrm{Tr}_2(s_1,s_2)}\underbrace{\mathrm{tr}\,\{(\tfrac{1}{2}\mathcal{T}^{t_1})^+m_0\mathcal{T}^{t_2}\}}_{=m_0\delta^{t_1 t_2}}\underbrace{\mathrm{tr}\,\{(\tfrac{1}{2}\mathcal{L}^{l_1})^+\mathcal{L}^{l_2}\}}_{=\delta^{l_1 l_2}}. \tag{46}$$

With these traces it follows from (44) with $H=(s\,t\,l)$:

$$\mathscr{H}_0 = \int d^3z\, b^{s_1 t l}(z)\,[-i\,\mathrm{Tr}_1(s_1,s_2,k)\,\partial_k + \bar{m}\,\mathrm{Tr}_2(s_1,s_2)]\,\frac{\delta}{\delta b^{s_2 t l}}(z) \tag{47}$$

with $\bar{m}=2m_0$. (Note: $(s_{1,2},t,l)\in\mathscr{A}\mathscr{s}$.)

Now we consider the second term of (41). With the definitions from (13) for $W_{I_1 I_2 I_3 I_4}$ and $F_{I_2 I}$ and $b_{H_1}(\boldsymbol{k}_1)$ according to (42) this term yields

$$\mathscr{H}_1 = 3\chi(0)\,g_H\,\mathrm{tr}\{w_H\,\mathscr{C}^{H_2}\}\int d^3k_2\,d^3z\,d^3u\,b^{H_1}(z)\,\mathrm{tr}\{\mathscr{R}_{H_1}^T\,\boldsymbol{u}^H\,F(\boldsymbol{u})\}\,\chi^*(\boldsymbol{u})\,e^{i\boldsymbol{k}_2\cdot\boldsymbol{u}/2}\,e^{i\boldsymbol{k}_2\cdot\boldsymbol{z}}\,\frac{\delta}{\delta b^{H_2}}(\boldsymbol{k}_2).$$

With (32) follows

$$\mathscr{H}_1 = -48\chi(0)\,g_{H_2}\int d^3k_2\,d^3z\,b^{H_1}(z)\underbrace{\int d^3u\,\mathrm{tr}\{\mathscr{R}_{H_1}^T\,\boldsymbol{u}^{H_2}\,F(\boldsymbol{u})\}\,\chi^*(\boldsymbol{u})\,e^{i\boldsymbol{k}_2\cdot\boldsymbol{u}/2}}_{=:\,\mathrm{Tr}_3(H_1 H_2|\boldsymbol{k}_2)}\,e^{i\boldsymbol{k}_2\cdot\boldsymbol{z}}\,\frac{\delta}{\delta b^{H_2}}(\boldsymbol{k}_2). \tag{48}$$

According to [4, 5] we use the low energy approximation

$$\mathrm{Tr}_3(H_1 H_2|\boldsymbol{k}) \simeq \mathrm{Tr}_3(H_1 H_2) = c_f\,\mathrm{tr}\{\mathscr{R}_{H_1}^T\,\boldsymbol{u}^{H_2}\,C\,c_\tau\,c_\lambda\}. \tag{49}$$

The constant c_f depends on $\chi^*(0)$ and the behaviour of the propagator near the origin. Now the $\boldsymbol{k}_2$-dependence of $\mathrm{Tr}_3(H_1 H_2|\boldsymbol{k}_2)$ is neglected and with (42) results

$$\mathscr{H}_1 = -48\chi(0)\,g_{H_2}\int d^3z\,b^{H_1}(z)\,\mathrm{Tr}_3(H_1 H_2)\,\frac{\delta}{\delta b^{H_2}}(z). \tag{50}$$

With (13) and (33) one finds for $\mathrm{Tr}_3(H_1 H_2)$:

$$\begin{aligned}\mathrm{Tr}_3(H_1 H_2) &= \mathrm{Tr}_3(s_1 t_1 l_1 s_2 t_2 l_2) = -c_f\,\eta_{s_2}\,\mathrm{tr}\{\mathscr{R}_{s_1 t_1 l_1}^T\,\gamma^0\,\mathscr{C}^{s_2 t_2 l_2}\}\\ &= -c_f\,\eta_{s_2}\underbrace{\mathrm{tr}\{(\tfrac{1}{4}\mathscr{S}^{s_1})^+\,\gamma^0\,\mathscr{S}^{s_2}\}}_{=\,\mathrm{Tr}_2(s_1,s_2)}\,\underbrace{\mathrm{tr}\{(\tfrac{1}{2}\mathscr{T}^{t_1})^+\,\mathscr{T}^{t_2}\}}_{=\,\delta^{t_1 t_2}}\,\underbrace{\mathrm{tr}\{(\tfrac{1}{2}\mathscr{L}^{l_1})^+\,\mathscr{L}^{l_2}\}}_{=\,\delta^{l_1 l_2}},\end{aligned} \tag{51}$$

i.e. up to a factor the trace $\mathrm{Tr}_2(H_1 H_2)$ from (46). With (51) finally it follows from (50):

$$\mathscr{H}_1 = c_1\,g_{s_2 t l}^*\int d^3z\,b^{s_1 t l}(z)\,\mathrm{Tr}_2(s_1 s_2)\,\frac{\delta}{\delta b^{s_2 t l}}(z) \tag{52}$$

with $c_1 := 48\,c_f\,\chi(0)$ and $g_{stl}^* := \eta_s\,g_{stl}$. Combining $\mathscr{H}_0$ from (47) and $\mathscr{H}_1$ from (52) we find for the kinemtic part of the bosonic Hamiltonian:

$$\mathscr{H}_{\mathrm{kin}} = \int d^3z\,b^{s_1 t l}(z)\,\{-i\,\mathrm{Tr}_1(s_1,s_2,k)\,\partial_k + \mathrm{Tr}_2(s_1 s_2)(\bar{m}+\Delta_{s_2 t l})\}\,\frac{\delta}{\delta b^{s_2 t l}}(z) \tag{53}$$

with $\Delta_{stl} := c_1\,g_{stl}^*$.

The term $\mathscr{H}_{bb}$ from (40) describing the boson-boson coupling leads with (35) to

$$\begin{aligned}\mathscr{H}_{bb} = -6\chi(0)\,g_H\,\mathrm{tr}\{w_H\,\mathscr{C}^{H_1}\}\int d^3k_1\,d^3k_2\,d^3k_3\,d^3z\,e^{-i\boldsymbol{k}_3\cdot\boldsymbol{z}}\,b^{H_3}(\boldsymbol{k}_3)\,\mathrm{tr}\{\boldsymbol{u}^H\,\mathscr{C}^{H_2}\,\mathscr{R}_{H_3}^T\}\\ \cdot\,e^{i\boldsymbol{k}_2\cdot\boldsymbol{z}}\,\frac{\delta}{\delta b^{H_2}}(\boldsymbol{k}_2)\underbrace{\int d^3u\,\chi^*(\boldsymbol{u})\,\chi(\boldsymbol{u})\,e^{i\boldsymbol{k}_1\cdot\boldsymbol{u}/2}}_{J(\boldsymbol{k}_1)}\,e^{i\boldsymbol{k}_1\cdot\boldsymbol{z}}\,\frac{\delta}{\delta b^{H_1}}(\boldsymbol{k}_1).\end{aligned} \tag{54}$$

In the low energy approximation the integral $J(\boldsymbol{k}_1)$ can be approximated by a constant $c\simeq 1$, neglecting the dependence from $\boldsymbol{k}_1$. Then with (42) and (32) it results from (54):

$$\mathscr{H}_{bb} = 96\,c\,\chi(0)\,g_{H_1}\int d^3z\,b^{H_3}(z)\underbrace{\mathrm{tr}\{\boldsymbol{u}^{H_1}\,\mathscr{C}^{H_2}\,\mathscr{R}_{H_3}^T\}}_{=:\,\mathrm{Tr}_4(H_1 H_2 H_3)}\,\frac{\delta}{\delta b^{H_2}}(z)\,\frac{\delta}{\delta b^{H_1}}(z). \tag{55}$$

The trace reads with (13) and $H=(s\,t\,l)$:

$$\begin{aligned}\mathrm{Tr}_4(H_1,H_2,H_3) &= -\,\mathrm{tr}\,\{\gamma^0\,\boldsymbol{v}^{H_1}\,\mathscr{C}^{H_2}\,\mathscr{R}^T_{H_3}\}\\ &= -\,\mathrm{tr}\,\{\gamma^0\,\Gamma^{s_1}\,(\mathscr{S}^{s_2}\tfrac{1}{4}\,\mathscr{S}^{s_3})^+\}\;\mathrm{tr}\,\{\tau^{t_1}\,\mathscr{T}^{t_2}\,\tfrac{1}{2}(\mathscr{T}^{t_3})^+\}\;\mathrm{tr}\,\{\lambda^{l_1}\,\mathscr{L}^{l_2}\,\tfrac{1}{2}(\mathscr{L}^{l_3})^+\}\\ &=:-\,\mathrm{Tr}_4(s_1\,s_2\,s_3)\,\mathrm{Tr}_4(t_1\,t_2\,t_3)\,\mathrm{Tr}_4(l_1\,l_2\,l_3).\end{aligned} \tag{56}$$

Thus we find for $\mathscr{H}_{bb}$ the result

$$\mathscr{H}_{bb}=c_{bb}\,g_{s_1t_1l_1}\int \mathrm{d}^3z\; b^{s_3t_3l_3}(z)\,\mathrm{Tr}_4(s_1\,s_2\,s_3)\,\mathrm{Tr}_4(t_1\,t_2\,t_3)\,\mathrm{Tr}_4(l_1\,l_2\,l_3)\,\frac{\delta}{\delta b^{s_2t_2l_2}}(z)\,\frac{\delta}{\delta b^{s_1t_1l_1}}(z) \tag{57}$$

with $c_{bb}:=-96\,c\,\chi(0)$. For the isospin trace it follows with (30):

$$\mathrm{Tr}_4(t_1\,t_2\,t_3)=\tfrac{1}{2}\,\mathrm{tr}\,\{\tau^{t_1}\,\tau^{t_2}\,\tau^{t_3}\}=i\,\varepsilon_{t_1t_2t_3}+\delta_{t_1t_2}\,\delta_{0t_3}+\delta_{0t_1}\,\delta_{t_2t_3/t_2\neq 0}+\delta_{0t_2}\,\delta_{t_1t_3/t_1\neq 0}\,. \tag{58}$$

For the superspin trace one analogously finds with (31):

$$\mathrm{Tr}_4(l_1\,l_2\,l_3)=\tfrac{1}{2}\,\mathrm{tr}\,\{\lambda^{l_1}\,\lambda^{l_2}\,\lambda^{l_3}\}=i\,\varepsilon_{l_1l_2l_3}+\delta_{l_1l_2}\,\delta_{0l_3}+\delta_{0l_1}\,\delta_{l_2l_3/l_2\neq 0}+\delta_{0l_2}\,\delta_{l_1l_3/l_1\neq 0}\,. \tag{59}$$

The Dirac traces $\mathrm{Tr}_4(s_1\,s_2\,s_3)=\frac{1}{4}\mathrm{tr}\,\{\gamma^0\,\Gamma^{s_1}\,\mathscr{S}^{s_2}(\mathscr{S}^{s_3})^+\}$ can be evaluated using (29) and the properties of the γ matrices. For brevity we don't write them down explicitly.

5. Boson-Fermion Coupling

The coupling between bosons and fermions is described by the following part of the functional Hamiltonian (22):

$$\begin{aligned}\mathscr{H}_{(f-b)}&=-\,6\,j_{I_1}\,W_{I_1I_2I_3I_4}\,\mathscr{C}^{I_4I_3}_K\,\frac{\delta}{\delta j_{I_2}}\,\frac{\delta}{\delta b_K}+3\,b_K\,W_{I_1I_2I_3I_4}\,F_{I_2I}\,\mathscr{R}^K_{I_1I}\,\frac{\delta}{\delta j_{I_3}}\,\frac{\delta}{\delta j_{I_4}}\\ &=:\mathscr{H}_{bf}+\mathscr{H}_{fb}.\end{aligned} \tag{60}$$

The first term leads to the coupling of the bosons in the fermionic field equation while the second describes the coupling of the fermionic current in the bosonic field equations. We first consider $\mathscr{H}_{bf}$. Integration yields

$$\mathscr{H}_{bf}=-\,3\,g_H\,\mathrm{tr}\,\{\boldsymbol{w}_H\,\mathscr{C}^{H'}\}\int \mathrm{d}^3k\,\mathrm{d}^3x_1\,j_{Z_1}(\boldsymbol{x}_1)\,\boldsymbol{u}^H_{Z_1Z_2}\,\frac{\delta}{\delta j_{Z_2}}(\boldsymbol{x}_1)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}_1}\,\chi(0|\boldsymbol{k})\,\frac{\delta}{\delta b^{H'}}(\boldsymbol{k}). \tag{61}$$

With the low energy approximation (35), the Fouriuer transform (42) of $\delta/\delta b_{H'}(\boldsymbol{k})$, and the orthogonality relation (32) it results

$$\mathscr{H}_{bf}=-\,c_{bf}\,g_{stl}\int \mathrm{d}^3x\left(j^T(\boldsymbol{x})\,\boldsymbol{u}^{stl}\,\frac{\delta}{\delta j}(\boldsymbol{x})\right)\frac{\delta}{\delta b^{stl}}(\boldsymbol{x}) \tag{62}$$

with the abbreviation $c_{bf}=-48\,\chi(0)$. The second term in (60) reads

$$\begin{aligned}\mathscr{H}_{fb}=3\int &\mathrm{d}^3k\,\mathrm{d}^3x\,\mathrm{d}^3x_1\,\mathrm{d}^3x_2\,\mathrm{d}^3x_3\,\mathrm{d}^3x_4\,b_{H_1}(\boldsymbol{k})\\ &\cdot[\tfrac{1}{2}\,g_{H_2}(\boldsymbol{u}_{H_2})_{Z_1Z_2}\,\boldsymbol{w}^{H_2}_{Z_3Z_4}\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_3)\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_4)]\\ &\cdot F_{Z_2Z}(\boldsymbol{x}_2-\boldsymbol{x})\,[e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\boldsymbol{x}_1)/2}\,\chi^*(\boldsymbol{x}_1-\boldsymbol{x}|\boldsymbol{k})\,\mathscr{R}^{H_1}_{Z_1Z}]\,\frac{\delta}{\delta j_{Z_3}}(\boldsymbol{x}_3)\,\frac{\delta}{\delta j_{Z_4}}(\boldsymbol{x}_4).\end{aligned} \tag{63}$$

We integrate over $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_4$ and get with $\boldsymbol{z}=\boldsymbol{x}_1$, $\boldsymbol{u}=\boldsymbol{x}_1-\boldsymbol{x}$:

$$\mathscr{H}_{fb}=\tfrac{3}{2}\,g_{H_2}\int \mathrm{d}^3k\,\mathrm{d}^3z\,b^{H_1}(\boldsymbol{k})\,\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{z}}\,\frac{\delta}{\delta j_{Z_3}}(\boldsymbol{z})\,(\boldsymbol{w}_{H_2})_{Z_3Z_4}\,\frac{\delta}{\delta j_{Z_4}}(\boldsymbol{z})\,\underbrace{\int \mathrm{d}^3u\,\mathrm{tr}\,\{\mathscr{R}^T_{H_1}\,\boldsymbol{u}^{H_2}\,F(\boldsymbol{u})\}\,e^{i\boldsymbol{k}\cdot\boldsymbol{u}/2}\,\chi^*(\boldsymbol{u}|\boldsymbol{k})}_{=\,\mathrm{Tr}_3(H_1\,H|\boldsymbol{k})}. \tag{64}$$

Here exactly the trace (48) appears. Thus, applying the low energy approximation (49) and inserting the expression (51) we find

$$\mathscr{H}_{fb} = c_{fb}\, g^{*}_{s_2 t l} \int d^3 t\, b^{s_1 t l}(z)\, \mathrm{Tr}_2(s_1 s_2) \left(\frac{\delta^T}{\delta j}(z)\, w_{s_2 t l} \frac{\delta}{\delta j}(z) \right) \tag{65}$$

with $c_{fb} = -\frac{3}{2} c_f$ and $g^{*}_{stl} = \eta_s g_{stl}$.

Combining the terms (53), (57), (62), (65), and the pure fermionic part from (22) we find for the functional boson-fermion Hamiltonian (22):

$$\begin{aligned} \tilde{\mathscr{H}}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b}\right] = \int d^3 z \Big\{ & b^{s_1 t l}(z)\left[-i\,\mathrm{Tr}_1(s_1, s_2, k)\,\partial_k + \mathrm{Tr}_2(s_1 s_2)(\bar{m} + \Delta_{s_2 t l})\right] \frac{\delta}{\delta b^{s_2 t l}}(z) \\ & + c_{bb}\, g_{s_1 t_1 l_1}\, b^{s_3 t_3 l_3}(z)\, \mathrm{Tr}_4(s_1 s_2 s_3)\, \mathrm{Tr}_4(t_1 t_2 t_3)\, \mathrm{Tr}_4(l_1 l_2 l_3) \frac{\delta}{\delta b^{s_2 t_2 l_2}}(z) \frac{\delta}{\delta b^{s_1 t_1 l_1}}(z) \\ & + c_{fb}\, g^{*}_{s_2 t l}\, b^{s_1 t l}(z)\, \mathrm{Tr}_2(s_1 s_2) \left(\frac{\delta^T}{\delta j}(z)\, w_{s_2 t l} \frac{\delta}{\delta j}(z) \right) \Big\} \\ + \int d^3 x \Big\{ & j^T(x)\left[\Gamma^k \partial_k + M\right] \frac{\delta}{\delta j}(x) + \frac{1}{2} g_{stl} \left(j^T(x)\, u_{stl} \frac{\delta}{\delta j}(x) \right) \left(\frac{\delta^T}{\delta j}(x)\, w^{stl} \frac{\delta}{\delta j}(x) \right) \\ & - c_{bf}\, g_{stl} \left(j^T(x)\, u^{stl} \frac{\delta}{\delta j}(x) \right) \frac{\delta}{\delta b^{stl}}(x) \Big\}. \end{aligned} \tag{66}$$

This expression represents the effective boson-fermion coupling theory on the functional level.

6. Effective Field Equations

In order to derive field equations corresponding to the functional equation (21) with $\tilde{\mathscr{H}}\left[j, \frac{\delta}{\delta j}, b, \frac{\delta}{\delta b}\right]$ from (66) we apply the projections (24) and (25). For the spinor field this yields the equation

$$i\,\partial_0 \psi(x) = \left[\Gamma^k \partial_k + M\right] \psi(x) - \tfrac{1}{2} g_{stl}\, u_{stl}\, \psi(x) (\psi^T(x)\, w^{stl} \psi(x)) - \hat{c}_{bf}\, g_{stl}\, u^{stl} \psi(x)\, \Phi_{stl}(x) \tag{67}$$

(with $\hat{c}_{bf} = i c_{bf}$), while the boson field equation is given by

$$\begin{aligned} i\,\partial_0 \Phi_{s_1 t_1 l_1}(x) = & \left[-i\,\mathrm{Tr}_1(s_1, s_2, k)\,\partial_k + \mathrm{Tr}_2(s_1 s_2)(\bar{m} + \Delta_{s_2 t_1 l_1})\right] \Phi_{s_2 t_1 l_1}(x) \\ & + \hat{c}_{bb}\, g_{s_3 t_3 l_3}\, \mathrm{Tr}_4(s_3 s_2 s_1)\, \mathrm{Tr}_4(t_3 t_2 t_1)\, \mathrm{Tr}_4(l_3 l_2 l_1)\, \Phi_{s_2 t_2 l_2}(x)\, \Phi_{s_3 t_3 l_3}(x) \\ & + i\, c_{fb}\, g^{*}_{s_2 t_1 l_1}\, \mathrm{Tr}_2(s_1 s_2)\, \psi^T(x)\, w_{s_2 t_1 l_1} \psi(x) \end{aligned} \tag{68}$$

($\hat{c}_{bb} = i c_{bb}$). Note that these equations contain all possible couplings for the set of bosonic fields characterized by (38). For the explicit field equations the isospin and superspin traces (58) and (59) and the Dirac-spin traces have to be inserted. This yields rather complicated expressions describing a coupling theory with arbitrary coupling structure.

To be more transparent we will restrict the bosonic fields to those with fermion number 0 (fermion-antifermion bound states). These states are characterized by superspin indices $l = 0$ or $l = 3$ (depending on the spin-isospin symmetry). Herein isosinglets are contained for $t = 0$ and isotriplets for $t = 1, 2, 3$. For brevity we introduce the short notations $A_\mu := A^{03}_\mu \ldots$ for the singlets and $\boldsymbol{A}_\mu := (A^{i0}_\mu) \ldots$ for the triplets. Then from the fermionic equation (67) which contains the field equations for the spinor field ψ and for the charge conjugated field ψ^G it follows for the spinor field equation

$$\begin{aligned} i\,\partial_0 = & -i\gamma^0 \gamma^k \partial_k \psi + m_0 \gamma^0 \psi + g_h \gamma^0 v_h \psi(x) (\bar{\psi}(x)\, v^h \psi(x)) + \bar{g}_h \gamma^0 v_h G \bar{\psi}^T(x) (\psi^T(x) G v^h \psi(x)) \\ & + \hat{c}_{bf} \gamma^0 [g_{Vs} \gamma^\mu \psi A_\mu + 2 g_{Ts} \Sigma^{0k} \psi E_k + 2 g_{Ts} \Sigma^k \psi B_k + g_{As} \gamma_5 \gamma^\mu \psi G_\mu + g_{Ps} i \gamma_5 \psi \phi + g_{Ss} \psi Z \\ & + g_{Vv} \gamma^\mu \boldsymbol{\tau} \psi \cdot \boldsymbol{A}_\mu + 2 g_{Tv} \Sigma^{0k} \boldsymbol{\tau} \psi \cdot \boldsymbol{E}_k + 2 g_{Tv} \Sigma^k \boldsymbol{\tau} \psi \cdot \boldsymbol{B}_k \\ & + g_{Av} \gamma_5 \gamma^\mu \boldsymbol{\tau} \psi \cdot \boldsymbol{G}_\mu + g_{Pv} i \gamma_5 \boldsymbol{\tau} \psi \cdot \boldsymbol{\phi} + g_{Sv} \boldsymbol{\tau} \psi \cdot \boldsymbol{Z}]. \end{aligned} \tag{69}$$

This equation corresponds to the original field equation (in energy representation) with additional couplings to the bosonic fields.

For states with fermion number 0, the bosonic equations (68) lead by evaluation of the traces $\mathrm{Tr}_4(l_3 l_2 l_1)$, $\mathrm{Tr}_4(t_3 t_2 t_1)$, $\mathrm{Tr}_4(s_3 s_2 s_1)$, and $\mathrm{Tr}_2(s_1 s_2)$ to field equations for isosinglet and isotriplet states. The isosinglet equations read

$$\begin{aligned}
\partial_0 A^0 &= -\partial_k A^k,\\
\partial_0 A^k &= -\partial_k A^0 + (\bar{m} + 2\Delta_{Ts}) E^k + 2\hat{c}_{fb} g_{Ts} \bar{\psi} \Sigma^{0k} \psi,\\
\partial_0 E^k &= -\varepsilon_{klm} \partial_l B^m - (\bar{m} + \Delta_{Vs}) A^k - \hat{c}_{fb} g_{Vs} \bar{\psi} \gamma^k \psi,\\
\partial_0 B^k &= -\varepsilon_{klm} \partial_l E^m,\\
\partial_0 G^0 &= -\partial_k G^k - (\bar{m} + \Delta_{Ps}) \phi + \hat{c}_{fb} g_{Ps} \bar{\psi} i \gamma_5 \psi,\\
\partial_0 G^k &= -\partial_k G^0,\\
\partial_0 \phi &= (\bar{m} + \Delta_{As}) G^0 - \hat{c}_{fb} g_{As} \bar{\psi} \gamma_5 \gamma^0 \psi,\\
\partial_0 Z &= 0
\end{aligned} \tag{70}$$

($\hat{c}_{fb} = 2 i c_{fb}$), where additional couplings to the isotriplets have been omitted for brevity. The triplet equations are given by

$$\begin{aligned}
\partial_0 \boldsymbol{A}^0 &= -\partial_k \boldsymbol{A}^k,\\
\partial_0 \boldsymbol{A}^k &= -\partial_k \boldsymbol{A}^0 + (\bar{m} + 2\Delta_{Tv}) \boldsymbol{E}^k,\\
&\quad + 2\hat{c}_{bb}\{g_{Vv} \boldsymbol{A}^0 \times \boldsymbol{A}^k + 2 g_{Tv} \varepsilon_{klm} \boldsymbol{E}^l \times \boldsymbol{B}^m + g_{Av} \boldsymbol{G}^0 \times \boldsymbol{G}^k\} + 2\hat{c}_{fb} g_{Tv} \bar{\psi} \Sigma^{0k} \boldsymbol{\tau} \psi,\\
\partial_0 \boldsymbol{E}^k &= -\varepsilon_{klm} \partial_l \boldsymbol{B}^m - (\bar{m} + \Delta_{Vv}) \boldsymbol{A}^k\\
&\quad + \hat{c}_{bb}\{(g_{Vv} + 2 g_{Tv})(\boldsymbol{A}^0 \times \boldsymbol{E}^k - \varepsilon_{klm} \boldsymbol{A}^l \times \boldsymbol{B}^m) - (g_{Av} + g_{Pv}) \boldsymbol{\phi} \times \boldsymbol{G}^k\} - \hat{c}_{fb} g_{Vv} \bar{\psi} \gamma^k \boldsymbol{\tau} \psi,\\
\partial_0 \boldsymbol{B}^k &= \varepsilon_{klm} \partial_l \boldsymbol{E}^m + \hat{c}_{bb}\{(g_{Vv} - 2 g_{Tv})(\boldsymbol{A}^0 \times \boldsymbol{B}^k + \varepsilon_{klm} \boldsymbol{A}^l \times \boldsymbol{E}^m) + (g_{Av} + g_{Sv}) \boldsymbol{Z} \times \boldsymbol{G}^k,\\
\partial_0 \boldsymbol{G}^0 &= -\partial_k \boldsymbol{G}^k - (\bar{m} + \Delta_{Pv}) \boldsymbol{\phi} + \hat{c}_{bb}(g_{Vv} - g_{Av}) \boldsymbol{A}_\mu \times \boldsymbol{G}^\mu + \hat{c}_{fb} g_{Pv} \bar{\psi} i \gamma_5 \boldsymbol{\tau} \psi,\\
\partial_0 \boldsymbol{G}^k &= -\partial_k \boldsymbol{G}^0 + \hat{c}_{bb}\{(g_{Vv} + g_{Av})(\boldsymbol{A}^0 \times \boldsymbol{G}^k - \boldsymbol{A}^k \times \boldsymbol{G}^0)\\
&\qquad + (2 g_{Tv} + g_{Pv}) \boldsymbol{E}^k \times \boldsymbol{\phi} + (2 g_{Tv} - g_{Sv}) \boldsymbol{B}^k \times \boldsymbol{Z}\},\\
\partial_0 \boldsymbol{\phi} &= (\bar{m} + \Delta_{Av}) \boldsymbol{G}_0 + \hat{c}_{bb}\{(g_{Vv} - g_{Pv}) \boldsymbol{A}^0 \times \boldsymbol{\phi} + (2 g_{Tv} - g_{Av}) \boldsymbol{E}^k + \boldsymbol{G}^k\} - \hat{c}_{fb} g_{Av} \bar{\psi} \gamma_5 \gamma^0 \boldsymbol{\tau} \psi,\\
\partial_0 \boldsymbol{Z} &= \hat{c}_{bb}\{(g_{Vv} - g_{Sv}) \boldsymbol{A}^0 \times \boldsymbol{Z} + (2 g_{Tv} - g_{Av}) \boldsymbol{B}^k \times \boldsymbol{G}^k\}
\end{aligned} \tag{71}$$

(omitting couplings to the isosinglets). Equations (69), (70), and (71) describe (in energy representation) a coupling theory involving the spinor field ψ, isoscalar and isovector spin-1 fields (A_μ, $F_{\mu\nu}$) and ($\boldsymbol{A}_\mu$, $\boldsymbol{F}_{\mu\nu}$) as well as isoscalar and isovector spin-0 fields (ϕ, G_μ) and ($\boldsymbol{\phi}$, $\boldsymbol{G}_\mu$). (The scalar quantities Z and $\boldsymbol{Z}$ are of no further importance.) The coupling structure depends on the coupling structure of the underlying spinorial self-interaction.

7. Phenomenological Equations for Gauge Coupling

For the comparison with (69), (70), and (71) we consider the corresponding phenomenological theory consisting of a complex spinor field $\psi_{\alpha A}(x)$, $A = 1, 2$ ($SU(2)$ isodoublet), a real pseudoscalar field $\phi^t(x)$ and a real vector field $A^t_\mu(x)$, the bosonic fields (characterized by the index $t = 0, \ldots, 3$) containing isosinglets $\phi := \phi^0$ resp. $A_\mu := A^\mu_0$ and isotriplets $\boldsymbol{\phi} := (\phi^i)$ resp. $\boldsymbol{A}_\mu := (A^i_\mu)$, $i = 1, 2, 3$.

In conventional coupling theory a variety of possible couplings between the fields can be considered restricted by (global) symmetry requirements. A more systematic treatment of interactions is provided by *local* gauge invariance. In the standard model of Glashow [23], Weinberg [24], and Salam [25] electroweak interactions are described by a Yang-Mills gauge theory. As we are interested in possible substructure explanations of the standard theory we consider gauge interactions to see whether these are contained in our effective field equations. In this context the vector fields $A_\mu^t(x)$ correspond to the electroweak gauge bosons, the spinor field $\psi(x)$ to a fermionic doublet, and the scalar field might be related to a possible composite Higgs field.

If the couplings are assumed to be fixed by local $U(1)\times SU(2)$ gauge invariance, the coupling theory is described by the Lagrangian

$$\begin{aligned}\mathcal{L}_{PT} &= \bar\psi i\gamma^\mu D_\mu\psi + (\partial_\mu\phi)(\partial^\mu\phi) + (D_\mu\boldsymbol{\phi})\cdot(D^\mu\boldsymbol{\phi})\\ &\quad -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu}\\ &= \bar\psi i\gamma^\mu\partial_\mu\psi + g_s\bar\psi\gamma^\mu\psi A_\mu + g_v\bar\psi\gamma^\mu\boldsymbol{\tau}\phi\cdot\boldsymbol{A}_\mu\\ &\quad + (\partial_\mu\phi)(\partial^\mu\phi) + (\partial_\mu\boldsymbol{\phi})\cdot(\partial^\mu\boldsymbol{\phi})\\ &\quad + 2g_v(\boldsymbol{A}_\mu\times\boldsymbol{\phi})\cdot\partial^\mu\boldsymbol{\phi} + g_v^2(\boldsymbol{A}_\mu\times\boldsymbol{\phi})\cdot(\boldsymbol{A}^\mu\times\boldsymbol{\phi})\\ &\quad -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu}\end{aligned} \tag{72}$$

with the covariant derivatives $D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + g_v\boldsymbol{A}_\mu\times\boldsymbol{\phi}$ for the scalar isotriplet field and $D_\mu\psi = (\partial_\mu - i g_t A_\mu^t\tau^t)\psi = (\partial_\mu - i g_s A_\mu - \tfrac{1}{2} i g_v\boldsymbol{A}_\mu\cdot\boldsymbol{\tau})\psi$ for the spinor field, the $U(1)$ and $SU(2)$ gauge coupling constants being denoted by g_s and g_v. The field strengths are given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for the isosinglet and $\boldsymbol{F}_{\mu\nu} = \partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu + g\boldsymbol{A}_\mu\times\boldsymbol{A}_\nu$ for the isotriplet. The gauge theory does not provide (direct) couplings between the scalar field and the spinor field. In the standard model these couplings are additionally introduced (Yukawa coupling). Furthermore in pure gauge theory no mass terms are present. In the standard theory masses result from couplings to the symmetry-breaking ground state. As in this paper we will not deal with the problem of mass generation and symmetry-breaking we simply add mass terms for the field to (72). Then the corresponding field equations read

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi &= m(\psi)\psi - g_t A_\mu^t\tau^t\gamma^\mu\psi\\ &= m(\psi)\psi - g_s A_\mu\gamma^\mu\psi - \tfrac{1}{2}g_v\boldsymbol{A}_\mu\cdot\boldsymbol{\tau}\gamma^\mu\psi\end{aligned} \tag{73}$$

for the spinor field,

$$\partial_\mu F^{\mu\nu} = -m^2(A)A^\nu - g_s\bar\psi\gamma^\nu\psi \tag{74}$$

for the vector isosinglet,

$$\begin{aligned}&\partial_\mu\boldsymbol{F}^{\mu\nu} + g_v\boldsymbol{A}_\mu\times\boldsymbol{F}^{\mu\nu}\\ &\quad = -m^2(A)\boldsymbol{A}^\nu - \tfrac{1}{2}g_v\bar\psi\gamma^\nu\boldsymbol{\tau}\psi - g_v(D^\nu\boldsymbol{\phi})\times\boldsymbol{\phi}\end{aligned} \tag{75}$$

for the vector isotriplet,

$$\partial_\mu\partial^\mu\phi = -m^2(\phi)\phi \tag{76}$$

for the scalar isosinglet, and

$$\begin{aligned}\partial_\mu\partial^\mu\boldsymbol{\phi} &= -m^2(\boldsymbol{\phi})\boldsymbol{\phi} - g_v(\partial_\mu\boldsymbol{A}^\mu)\times\boldsymbol{\phi}\\ &\quad -2g_v\boldsymbol{A}_\mu\times\partial^\mu\boldsymbol{\phi} - g_v^2\boldsymbol{A}_\mu\times(\boldsymbol{A}^\mu\times\boldsymbol{\phi})\end{aligned} \tag{77}$$

for the scalar isotriplet. In order to compare these equations with those derived by weak mapping, we rewrite them in an energy representation using first order time derivatives of the fields [26]. For the spinor field, it directly follows from (73):

$$\begin{aligned} i\partial_0\psi &= -i\gamma^0\gamma^k\partial_k\psi + m(\psi)\gamma^0\psi - g_s\gamma^0\gamma^\mu\psi A_\mu\\ &\quad -\tfrac{1}{2}g_v\gamma^0\gamma^\mu\boldsymbol{\tau}\psi\cdot\boldsymbol{A}_\mu.\end{aligned} \tag{78}$$

For the vector isosinglet A_μ it follows from (74) together with the field strength definitions and the notation $E^k := F^{0k}$, $B^k := -\tfrac{1}{2}\varepsilon_{ijk}F^{ij}$:

$$\partial_0 A^k = E^k - \partial_k A^0, \tag{79}$$

$$\partial_0 E^k = -\varepsilon_{klm}\partial_l B^m - m^2(A)A^k - g_s\bar\psi\gamma^k\psi, \tag{80}$$

$$\partial_0 B^k = \varepsilon_{klm}\partial_l E^m. \tag{81}$$

The isotriplet vector field $\boldsymbol{A}_\mu$ and the corresponding field strengths $\boldsymbol{E}^k := \boldsymbol{F}^{0k}$, $\boldsymbol{B}^k := -\tfrac{1}{2}\varepsilon_{ijk}\boldsymbol{F}^{ij}$ from (75) fulfil the following system:

$$\partial_0\boldsymbol{A}^k = \boldsymbol{E}^k - \partial_k\boldsymbol{A}^0 - g_v\boldsymbol{A}^0\times\boldsymbol{A}^k, \tag{82}$$

$$\begin{aligned}\partial_0\boldsymbol{E}^k &= -\varepsilon_{klm}(\partial_l\boldsymbol{B}^m - g_v\boldsymbol{A}^l\times\boldsymbol{B}^m) - g_v\boldsymbol{A}^0\times\boldsymbol{E}^k\\ &\quad -m^2(A)\boldsymbol{A}^k - \tfrac{1}{2}g_v\bar\psi\gamma^k\boldsymbol{\tau}\psi - g_v\boldsymbol{G}^k\times\boldsymbol{\phi},\end{aligned} \tag{83}$$

$$\partial_0\boldsymbol{B}^k = \varepsilon_{klm}(\partial_l\boldsymbol{E}^m - g_v\boldsymbol{A}^l\times\boldsymbol{E}^m) - g_v\boldsymbol{A}^0\times\boldsymbol{B}^k. \tag{84}$$

For the scalar isosinglet (76) the energy representation is given by

$$\partial_0 = G^0, \tag{85}$$

$$\partial_0 G^0 = -\partial_k G^k - m^2(\phi)\phi, \tag{86}$$

$$\partial_0 G^k = -\partial_k G^0. \tag{87}$$

Here the field G^μ serves as gradient field to ϕ. Finally, for the scalar isotriplet one finds

$$\partial_0\boldsymbol{\phi} = \boldsymbol{G}^0 - g_v\boldsymbol{A}^0\times\boldsymbol{\phi}, \tag{88}$$

$$\partial_0\boldsymbol{G}^0 = -\partial_k\boldsymbol{G}^k - m^2(\boldsymbol{\phi})\boldsymbol{\phi} - g_v\boldsymbol{A}_\mu\times\boldsymbol{G}^\mu, \tag{89}$$

$$\partial_0\boldsymbol{G}^k = -\partial_k\boldsymbol{G}^0 - g_v(\boldsymbol{E}^k\times\boldsymbol{\phi} + \boldsymbol{A}^k\times\boldsymbol{G}^0 - \boldsymbol{A}^0\times\boldsymbol{G}^k) \tag{90}$$

as energy representation of (77) (with the covariant gradient field $\boldsymbol{G}^\mu$).

8. Effective Field Equations for Vector Coupling

To compare the phenomenological gauge coupling theory with the effective field equations derived in the preceeding sections we consider the case of pure vector coupling, i.e. we put

$$g_{At} = g_{St} = g_{Pt} = g_{Tt} = 0 \tag{91}$$

in (69), (70), and (71). Additionally we assume $Z = \boldsymbol{Z} = 0$. Then it results for the spinor field equation (69) (disregarding the self-coupling):

$$\begin{aligned} i\,\partial_0 \psi &= -i\gamma^0\gamma^k\psi + m_0\gamma^0\psi \\ &\quad -\gamma^0 [g_{Vs}^{bf}\gamma^\mu\psi A_\mu + g_{Vv}^{bf}\gamma^\mu \boldsymbol{\tau}\psi\cdot \boldsymbol{A}_\mu] \end{aligned} \tag{92}$$

with $g_{Vs}^{bf} := -\hat{c}_{bf}\, g_{Vt}$. Introducing the rescalings

$$E_k^t \to \bar{m}^{-1} E_k^t,\quad B_k^t \to \bar{m}^{-1} B_k^t,\quad G_\mu^t \to \bar{m}^{-1} G_\mu^t \tag{93}$$

one finds for the singlet equations (70) in the case of pure vector coupling:

$$\partial_0 A^0 = -\partial_k A^k, \tag{94}$$

$$\partial_0 A^k = -\partial_k A^0 + E^k, \tag{95}$$

$$\partial_0 E^k = -\varepsilon_{klm}\partial_l B^m - \bar{m}(\bar{m} + \Delta_{Vs}) A^k - g_{Vs}^{bf}\bar{\psi}\gamma^k\psi, \tag{96}$$

$$\partial_0 B^k = \varepsilon_{klm}\partial_l E^m, \tag{97}$$

$$\partial_0 G^0 = -\partial_k G^k - \bar{m}^2\phi, \tag{98}$$

$$\partial_0 G^k = -\partial_k G^0, \tag{99}$$

$$\partial_0 \phi = G^0 \tag{100}$$

with the abbreviation $g_{Vs}^{bf} := \hat{c}_{bf}\, g_{Vs}\bar{m}$. From (71) it follows for the triplets:

$$\partial_0 \boldsymbol{A}^0 = -\partial_k \boldsymbol{A}^k, \tag{101}$$

$$\partial_0 \boldsymbol{A}^k = -\partial_k \boldsymbol{A}^0 + \boldsymbol{E}^k - 2g_{bb}\boldsymbol{A}^0\times\boldsymbol{A}^k, \tag{102}$$

$$\begin{aligned} \partial_0 \boldsymbol{E}^k &= -\varepsilon_{klm}\partial_l \boldsymbol{B}^m - \bar{m}(\bar{m}+\Delta_{Vv})\boldsymbol{A}^k \\ &\quad - g_{bb}\{\boldsymbol{A}^0\times\boldsymbol{E}^k - \varepsilon_{klm}\boldsymbol{A}^l\times\boldsymbol{B}^m\} - g_{Vv}^{bf}\bar{\psi}\gamma^k\boldsymbol{\tau}\psi, \end{aligned} \tag{103}$$

$$\partial_0 \boldsymbol{B}^k = \varepsilon_{klm}\partial_l \boldsymbol{E}^m - g_{bb}\{\boldsymbol{A}^0\times\boldsymbol{B}^k + \varepsilon_{klm}\boldsymbol{A}^l\times\boldsymbol{E}^m\}, \tag{104}$$

$$\partial_0 \boldsymbol{G}^0 = -\partial_k \boldsymbol{G}^k - \bar{m}^2\boldsymbol{\phi} - g_{bb}\boldsymbol{A}_\mu\times\boldsymbol{G}^\mu, \tag{105}$$

$$\partial_0 \boldsymbol{G}^k = -\partial_k \boldsymbol{G}^0 - g_{bb}\{\boldsymbol{A}^0\times\boldsymbol{G}^k - \boldsymbol{A}^k\times\boldsymbol{G}^0\}, \tag{106}$$

$$\partial_0 \boldsymbol{\phi} = \boldsymbol{G}^0 - g_{bb}\boldsymbol{A}^0\times\boldsymbol{\phi} \tag{107}$$

with the definitions $g_{bb} := -\hat{c}_{bb}\, g_{Vv}$ and $g_{Vv}^{bf} := \hat{c}_{fb}\, g_{Vv}\bar{m}$.

If one compares (92) with (78) one recognizes that the vector part of the spinor coupling yields the gauge-type coupling to the vector fields A_μ^t. The isoscalar and isovector coupling constants g_{Vs}^{bf} and g_{Vv}^{bf} correspond to the phenomenological constants g_s and $\frac{1}{2}g_v$ of the $U(1)$ and $SU(2)$ gauge theory. As in the weak mapping procedure the fermions were represented by elementary spinor fields the kinematical part of (92) corresponds to the original field equation.

The vector boson isosinglet system (94)–(97) is identical with the phenomenological form (79)–(81). Especially a vector boson mass $m^2(A) = \bar{m}(\bar{m} + \Delta_{Vs})$ can be observed which is determined by the constituting spinors and their binding energy. Remarkably, with equation (94) the Lorentz condition appears. The coupling to the spinorial current is characterized by the constant g_{Vs}^{fb} instead of g_s. The consistence condition $g_{Vs}^{fb} = g_{Vs}^{bf}$ can be achieved by renormalization [21]. The equations (98)–(100) for the scalar isosinglet (ϕ, G_μ) are identical with the equations (85)–(87) with $m^2(\phi) = \bar{m}^2$.

The non-abelian vector field equations (101)–(104) are (almost) equivalent to the phenomenological counterpart (82)–(84). In analogy to the abelian case, the gauge coupling to the fermions ($g_{Vs}^{fb} \triangleq \frac{1}{2}g_v$), a mass term with $m^2(\boldsymbol{A}) = \bar{m}(\bar{m} + \Delta_{Vv})$, and the Lorentz condition (101) appear. The bosonic self-interaction results with the identification $g_{bb} = g_v$ up to a factor 2 in the field strength definition ((102) and (82)). The kinetic part of the equations (105)–(107) for the scalar isotriplet is identical with the corresponding part from (88)–(90) with $m^2(\boldsymbol{\phi}) = \bar{m}^2$. The coupling to the vector field results (almost) with $g_{bb} = g_v$. In this case, however, the term $-g_v\boldsymbol{E}^k\times\boldsymbol{\phi}$ from (90) is missing. Additionally there is no coupling term of the scalar field in the $\boldsymbol{E}$-field equation (103).

For the interpretation of the resulting equations we first observe that the non-abelian field equations (101)–(107) result from the corresponding free equations (94)–(100) by the substitution $\partial_\mu \to D_\mu = \partial_\mu + g\boldsymbol{A}_\mu\times$. This may be more transparent in a covariant formulation. With $D_\mu = \partial_\mu + g\boldsymbol{A}_\mu\times$ the field equations (101)–(104) read

$$\begin{aligned} D_\mu \boldsymbol{F}^{\mu\mu} &= -m^2\boldsymbol{A}^\nu, \\ \boldsymbol{F}_{\mu\nu} &= D_\mu\boldsymbol{A}_\nu - D_\nu\boldsymbol{A}_\mu. \end{aligned}$$

The equations (105)–(107) can be completed to the covariant form

$$\begin{aligned} \boldsymbol{G}_\mu &= D_\mu\boldsymbol{\phi}, \\ D_\mu\boldsymbol{G}_\nu &= D_\nu\boldsymbol{G}_\mu, \\ D_\mu\boldsymbol{G}^\mu &= -m^2\boldsymbol{\phi}. \end{aligned}$$

The same result is achieved if in the Bargmann-Wiegner equations the ordinary derivative is replaced by the gauge-covariant derivative. Physically thus the coupling of the spinor field (and its composites) to an external vector field is described. If the external field is formally identified with the composite vector field (what was actually done in our calculations) the deviations mentioned above result. We therefore consider these deviations to have formal reasons rather than physical ones and postpone a conclusive discussion to a more sophisticated treatment of the mapping procedure in forthcoming publications. It should be noted, however, that for a massless vector field (achieved by $\bar{m} = -\Delta_V$) these deviations drop out in temporal gauge, which might be interpreted as a hint that this gauge is somehow distinguished by nature.

If besides the vector coupling other couplings are considered, a more or less complicated coupling structure depending on the coupling constants g_H results. For example in the subfermion model of Stumpf [3–7] the Fierz-antisymmetrized coupling is characterized by the coupling constants

$$g_{Vs} = g_{Vv} = -g_{As} = -g_{Av} = -\tfrac{1}{12} g, \quad g_{Ss} = g_{Ps} = \tfrac{1}{3} g. \tag{108}$$

If these coupling constants are inserted into (69), (70), and (71) the resulting equations contain besides the vector coupling an axialvector as well as scalar and pseudoscalar couplings. This coupling structure is made manifest in the spinor field equation. In the bosonic field equations besides the vector currents scalar currents appear. The scalar field masses are modified by axialvector and pseudoscalar coupling. The boson-boson couplings are relatively complicated. But if one considers only the vector boson triplet the same equations as for pure vector coupling result, because in this case only the vector part contributes. Thus in the Stumpf model the effective non-abelian gauge theory appears exactly as described in the preceeding section. This was already shown in [5, 13]. In the present work these investigations were extended to a general coupling theory by considering additionally the vector field singlet, the scalar singlet and triplet.

9. Conclucion

For a nonlinear spinor field model with arbitrary four-fermion coupling structure we have derived the set of all algebraic possible field equations (in a low energy approximation) for two-fermion bound boson states and their couplings to elementary fermions using the method of weak mapping. For fermion number 0 boson states and vector coupling the effective equations correspond to phenomenological equations with gauge couplings. This shows that the gauge theories of the standard model of electroweak interaction can in principle be reproduced from a deeper subfermion structure. For a realistic picture, however, the considerations have to be extended at least to three-fermion bound leptons and quarks, and the isospin symmetry breaking has to be taken into account. These questions are topics of further investigations. A first discussion of the broken symmetry case is described in [27].

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[1] R. Kerschner and H. Stumpf, Ann. Phys. Leipzig **48,** 56 (1991).
[2] W. Pfister and H. Stumpf, Z. Naturforsch. **46a,** 389 (1991).
[3] D. Großer, B. Hailer, L. Hornung, T. Lauxmann, and H. Stumpf, Z. Naturforsch. **38a,** 1056 (1983).
[4] H. Stumpf, Z. Naturforsch. **40a,** 14 (1985).
[5] H. Stumpf, Z. Naturforsch. **41a,** 683 (1986).
[6] H. Stumpf, Z. Naturforsch. **42a,** 213 (1987).
[7] H. Stumpf, Z. Naturforsch. **43a,** 345 (1988).
[8] W. Heisenberg, Introduction to the unifield theory of elementary particles, Intersc. Publ., London 1966.
[9] H. Stumpf, Z. Naturforsch. **40a,** 183 (1985).
[10] H. Stumpf, Z. Naturforsch. **40a,** 294 (1985).
[11] H. Stumpf, Z. Naturforsch. **40a,** 752 (1985).
[12] W. Pfister, Thesis, University of Tübingen, 1990.
[13] H. Stumpf and W. Pfister, Nuovo Cim. **105A,** 677 (1992).
[14] H. Stumpf, Z. Naturforsch. **41a,** 1399 (1986).
[15] H. P. Dürr and H. Saller, Phys. Rev. **D22,** 1176 (1980).
[16] H. Saller, Nuovo Cim. **67A,** 70 (1982).
[17] G. Grimm, The two-particle sector of generalized Bargmann-Wigner equations. To be published in Nuovo Cim. **A.**
[18] H. Stumpf, Z. Naturforsch. **24a,** 188 (1969).
[19] H. Stumpf and K. Scheerer, Z. Naturforsch. **30a,** 1361 (1975).

[20] H. Stumpf, New representation spaces of the Poincaré group and functional quantum theory. In: Groups, Systems and Many-Body Physics (P. Kramer and M. Dal Clin, eds.), Vieweg, Braunschweig 1980.
[21] W. Pfister, Nuovo Cim. **106 A,** 363 (1993).
[22] W. Pfister, M. Rosa, and H. Stumpf, Nuovo Cim. **102 A,** 1449 (1989).
[23] S. L. Glashow, Nucl. Phys. **22,** 579 (1961).
[24] S. Weinberg, Phys. Rev. Lett. **19,** 1264 (1967).
[25] A. Salam, Weak and electromagnetic interactions. In: Elementary particle physics (Nobel Symp. No. 8), pages 367–377 (N. Svartholm, ed.), Almquist & Wiksell, Stockholm 1968.
[26] G. Grimm, On Gauss's law in classical non-abelian gauge theory. Preprint, University of Tübingen, 1993.
[27] G. Grimm, Effective boson-fermion dynamics for subfermion models with symmetry breaking. In preparation.